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Periodic orbits around the collinear liberation points in the restricted three body problem when both the primaries are triaxial rigid bodies

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Abstract : Periodic orbits around the collinear liberation points in the restricted three body problem have been studied when both the primaries are triaxial rigid bodies. The effects of perturbations on the semi-axes of the earth and the moon in the respective periodic orbits have been studied. The Liapunov stability of each periodic solution has also been examined.

Keywords : Restricted three-body problem, triaxial rigid body, periodic orbits, Liapunov stability.

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1. Introduction

In our previous paper [1], we have studied the periodic orbits around the collinear liberation points in the restricted three body problem. In that paper, we assumed the smaller primary as a triaxial rigid body with its equatorial plane coincident with the plane of motion. The periodic orbits have been determined by taking different values of semi-axes of the triaxial rigid body. The Liapunov stability of each periodic solution has also been examined for $\mu = 0.00095$. In this paper, we wish to generalize the earlier problem by taking both the primaries as triaxial rigid bodies *viz.* the earth-moon system. In the present article, we continue the investigation using the value for $\mu = 0.01215$ for the mass parameter of the problem. This is a more realistic case of the restricted problem in celestial mechanics since it corresponds to the earth-moon – satellite system. The numerical study presented here is based on the same methods and techniques as the investigations mentioned in earlier paper. Regarding the numerical integration of the equation of motion, the search for periodic orbits, and the numerical determination of stability, we shall use the same techniques.

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In this article, we have studied the effect of small perturbations on the semi-axes of the primaries. We consider here five cases (see Table 1). The Liapunov stability of each periodic orbit is also examined.

Table 1. Parameters of earth and moon.

Cases	1	2	3	4	5
a_1	6400	6400	6400	6400	6400
a_2	6400	6390	6380	6370	6360
a_3	6400	6380	6360	6340	6320
a'_1	1750	1750	1750	1750	1750
a'_2	1750	1740	1730	1720	1710
a'_3	1750	1730	1710	1690	1670

a_1, a_2, a_3 and a'_1, a'_2, a'_3 are the semi-axes of the earth and the moon respectively.

2. Equation of motion and variation

In the usual barycentric, rotating and dimensionless coordinate system (X,Y), with the two main bodies having masses m_1 and m_2, the equations of motion of the third particle m_3 in the phase space (X_1, X_2, X_3, X_4) are

$$\dot{X}_i = f_i(X_1, \ldots\ldots, X_4), \qquad i = 1\ldots\ldots4, \tag{1}$$

with

$$f_1 = X_3,\ f_2 = X_4$$

$$\begin{aligned} f_3 = 2nX_4 + n^2 X_1 &- \frac{(1-\mu)(X_1-\mu)}{{}^3} - \frac{\mu(X_1+1-\mu)}{r_2^3} \\ &- \frac{3\mu(2\sigma_1-\sigma_2)(X_1+1-\mu)}{2r_2^5} \\ &+ \frac{15\mu(\sigma_1-\sigma_2)(X_1+1-\mu)X_2^2}{2r_2^7} \\ &- \frac{3(1-\mu)(2\sigma_1'-\sigma_2')(X_1-\mu)}{2r_1^5} \\ &+ \frac{15(1-\mu)(\sigma_1'-\sigma_2')(X_1-\mu)X_2^2}{2r_1^7}, \end{aligned}$$

$$\begin{aligned} f_4 = -2nX_3 + n^2 X_2 &- \frac{(1-\mu)X_2}{r_1^3} - \frac{\mu X_2}{r_2^3} \\ &- \frac{3\mu(2\sigma_1-\sigma_2)X_2}{2r_2^5} - \frac{3\mu(\sigma_1-\sigma_2)X_2}{2r_2^5} \\ &+ \frac{15\mu(\sigma_1-\sigma_2)X_2^3}{2r_2^7} - \frac{3(1-\mu)(2\sigma_1'-\sigma_2')X_2}{2r_1^5} \\ &- \frac{3(1-\mu)(\sigma_1'-\sigma_2')X_2}{2r_1^5} + \frac{15(1-\mu)(\sigma_1'-\sigma_2')X_2^3}{2r_1^7} \end{aligned}$$

where

$$X_1 = X,\ X_2 = Y,\ X_3 = \dot{X},\ X_4 = \dot{Y},$$
$$r_1^2 = (X_1-\mu)^2 + X_2^2,\ r_2^2 = (X_1+1-\mu)^2 + X_2^2,$$

$$\mu = \frac{m_2}{m_1+m_2} \le \frac{1}{2},$$

m_1, m_2 ($m_1 \ge m_2$) being the masses of the primaries,

$$\sigma_1 = \frac{a_1^2 - a_3^2}{5R^2} \quad \sigma_2 = \frac{a_2^2 - a_3^2}{5R^2},\ \sigma_1, \sigma_2 \lhd\lhd 1,$$

$$\frac{{}'^2 \quad {}'^2}{5R^2} \qquad \frac{{}'^2 \quad {}'^2}{5R^2}$$

R is the dimensional distance between the earth and the moon.

Here, we have taken only first order terms of $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$.

The mean motion n of the primaries is given by

$$n = 1 + \frac{3}{4}(2\sigma_1 - \sigma_2) + \frac{3}{4}(2\sigma_1' - \sigma_2').$$

The coordinates of the infinitesimal particle in phase space $X_1, \ldots\ldots, X_4$ depend uniquely, along any solution, on the initial conditions $(X_{01}, \ldots\ldots, X_{04})$ and the time t *i.e.* $X_i = X_i(X_{01}, \ldots\ldots, X_{04}, t)$, $i = 1 \ldots\ldots 4$. Their partial derivatives with respect to the initial conditions satisfying the equations of variation are

$$\frac{d}{dt}\left(\frac{\partial X_i}{\partial X_{oj}}\right) = \sum_{k=1}^{4} \frac{\partial f_i}{\partial X_k} \cdot \frac{\partial X_k}{\partial X_{oj}}, i, j = \quad ..4. \tag{2}$$

If we denote the variations $\frac{\partial X_i}{\partial X_{oj}}$ by v_{ij}, we can write these last equations more explicitly as follows :

$$\dot{v}_{ij} = v_{(i+2)j},\ i = 1, 2,\ j = 1, 2, 3, 4;$$
$$\dot{v}_{ij} = f_{i1}v_{1j} + f_{i2}v_{2j} + f_{i3}v_{3j} + f_{i4}v_{4j},$$
$$i = 3, 4, j = 1, 2, 3, 4;$$

where

$$f_{ij} = \frac{\partial f_i}{\partial X_{oj}},$$

$$\begin{aligned} f_{31} = &-Q + n^2 + R_1(X_1-\mu)^2 + R_2(X_1+1-\mu)^2 \\ &- R_3 + R_4(X_1+1-\mu)^2 + R_6 X_2^2 \\ &- R_7(X_1+1-\mu)^2 X_2^2 - R_8 + R_9(X_1-\mu)^2 \\ &+ R_{11}X_2^2 - R_{12}(X_1-\mu)^2 X_2^2, \end{aligned}$$

$$\begin{aligned} f_{32} = &-Q_1 X_2 + R_4(X_1+1-\mu)X_2 \\ &+ 2R_2(X_1+1-\mu)X_2 - R_7(X_1+1-\mu)X_2^3 \\ &+ R_9(X_1-\mu)X_2 + 2R_{11}(X_1-\mu)X_2 \\ &- R_{12}(X_1-\mu)X_2^3, \end{aligned}$$

$$f_{33} = 0,\ f_{34} = 2n,\ f_{41} = f_{32},$$

$$\begin{aligned} f_{42} = &-Q + n^2 + (R_1 + R_2)X_2^2 - R_3 + R_4 X_2^2 \\ &- R_5 + 5R_6 X_2^2 - R_7 X_2^4 - R_8 + R_9 X_2^2 - R_{10} \\ &+ 5R_{11}X_2^2 - R_{12}X_2^4, \end{aligned}$$

$$f_{43} = -2n,\ f_{44} = 0,$$

$$R_1 = \frac{3(1-\mu)}{r_1^5},\ R_2 = \frac{3\mu}{r_2^5},\ R_3 = \frac{R_2(2\sigma_1-\sigma_2)}{2},$$

$$R_4 = \frac{5R_3}{r_2^2},\ R_5 = R_2(\sigma_1-\sigma_2),\ R_6 = \frac{5R_5}{2r_2^2},$$

$$R_7=\frac{7R_6}{r_2^2},\quad R_8=\frac{R_1(2\sigma_1'-\sigma_2')}{2},\quad R_9=\frac{5R_8}{r_1^2},$$

$$R_{10}=R_1(\sigma_1'-\sigma_2'),\quad R_{11}=\frac{5R_{10}}{2r_1^2},\quad R_{12}=\frac{7R_{11}}{r_1^2}.$$

$$Q=\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3},$$
$$Q_1=-R_1(X_1-\mu)-R_2(X_1+1-\mu).$$

The stability parameters a, b, c and d as used by Markellos [2] are :

$$a=v_{11}+sv_{14},\quad b=v_{13},$$
$$c=v_{31}-2(1+s)v_{21}-s^2v_{24},$$
$$d=v_{33}-(2+s)v_{23},\quad s=s_c+s_\sigma,$$

where

$$s_c=\frac{-1}{X_{04}}\left[n^2X_{01}-\frac{1-\mu}{|X_{01}-\mu|(X_{01}-\mu)}\right.$$

$$|X_{01}-\mu+1|(X_{01}-\mu+1)$$

$$\frac{3\mu(2\sigma_1-\sigma_2)}{}$$
$$X_{04}\left[2|X_{01}-\mu+1|(X_{01}-\mu+1)^3\right.$$
$$\frac{3(1-\mu)(2\sigma_1'-\sigma_2')}{2|X_{01}-\mu|(X_{01}-\mu)^3}$$

3. Motion around the collinear equilibrium points

We have calculated the collinear equilibrium points by giving small perturbations in the semi-axes of the earth and the moon by taking $\mu=0.01215, R=384400$ km (see Table 2).

We know for collinear equilibrium points.

$$f_3-2nX_4=0,\quad X_2=0$$

i.e.

$$n^2X_1-\frac{(1-\mu)(X_1-\mu)}{|X_1-\mu|^3}-\frac{\mu(X_1+1-\mu)}{|X_1+1-\mu|^3}$$
$$-\frac{3\mu(2\sigma_1-\sigma_2)(X_1+1-\mu)}{2|X_1+1-\mu|^5}$$
$$-\frac{3(1-\mu)(2\sigma_1'-\sigma_2')(X_1-\mu)}{2|X_1-\mu|^5}=0.$$

The characteristic roots of these points are

$$\lambda_i=\pm\left(\frac{\lambda_{c1}}{\sqrt{2}}+\lambda_{\sigma1}\right),\quad i=1,2$$

$$\lambda_i=\pm\left(\frac{\lambda_{c2}}{\sqrt{2}}-\lambda_{\sigma2}\right),\quad i=3,4$$

$$\lambda_{c1}=[R-2+(9R^2-8R)^{1/2}]^{1/2},$$

$$\lambda_{c2}=[R-2-(9R^2-8R)^{1/2}]^{1/2},$$

$$\lambda_{\sigma1}=\frac{q_1-q_2-6(2\sigma_1-\sigma_2)-6(2\sigma_1'-\sigma_2')}{2\sqrt{2}\lambda_{c1}}+\frac{6(2-R)[(2\sigma_1-\sigma_2)+(2\sigma_1'-\sigma_2')]}{2\sqrt{2}(9R^2-8R)^{1/2}\lambda_{c1}}+\frac{q_2(4+3R)-q_1(4-3R)}{2\sqrt{2}(9R^2-8R)^{1/2}\lambda_{c1}},$$

$$\lambda_{\sigma2}=\frac{q_1-q_2-6(2\sigma_1-\sigma_2)-6(2\sigma_1'-\sigma_2')}{2\sqrt{2}\lambda_{c2}}+\frac{6(2-R)[(2\sigma_1-\sigma_2)+(2\sigma_1'-\sigma_2')]}{2\sqrt{2}(9R^2-8R)^{1/2}\lambda_{c2}}+\frac{q_2(4+3R)-q_1(4-3R)}{2\sqrt{2}(9R^2-8R)^{1/2}\lambda_{c2}},$$

Table 2. Collinear liberation points.

Case	1	2	3	4	5
σ_1	0				
σ_2	0	3.46×10^{-7}	$6..908\times10^{-7}$	1.0345×10^{-6}	1.377×10^{-6}
σ_3	0	1.728×10^{-7}	3.449×10^{-7}	5.161×10^{-7}	6.865×10^{-7}
σ'_1	0	9.42×10^{-8}	1.873×10^{-7}	2.794×10^{-7}	3.703×10^{-7}
σ'_2	0	4.697×10^{-8}	9.312×10^{-8}	1.385×10^{-7}	1.83×10^{-7}
L_1	−1.1556799131	−1.1556813891	−1.1556828602	−1.1556843277	−1.1556857907
L_2	−0.8369180073	−0.8369163553	−0.8369147088	−0.8369130663	−0.8369114289
L_3	1.0050624087	1.0050621301	1.0050618682	1.0050616129	1.00506136

and their corresponding angular frequencies are

$$\omega_{ci} = -\frac{\lambda_{ci}}{\sqrt{2}}, \quad i = 1, 2$$
$$\omega_{\sigma i} = -\lambda_{\sigma i}, \quad i = 1, 2$$

where

$$R = \frac{(1-\mu)}{|X_{Lj}-\mu|^3} + \frac{\mu}{|X_{Lj}+1-\mu|^3},$$

$$q_1 = \frac{3}{2}[(2\sigma_1 - \sigma_2) + (2\sigma_1' - \sigma_2')] + \frac{6\mu(2\sigma_1-\sigma_2)}{|X_{Lj}+1-\mu|^5} + \frac{6(1-\mu)(2\sigma'_1-\sigma'_2)}{|X_{Lj}-\mu|^5}$$

$$q_2 = \frac{-3}{2}[(2\sigma_1 - \sigma_2) + (2\sigma_1' - \sigma_2')] + \frac{3\mu(2\sigma_1-\sigma_2)}{2|X_{Lj}+1-\mu|^5} + \frac{3\mu(\sigma_1-\sigma_2)}{|X_{Lj}+1-\mu|^5} + \frac{3(1-\mu)(2\sigma'_1-\sigma'_2)}{2|X_{Lj}-\mu|^5} + \frac{3(1-\mu)(\sigma'_1-\sigma'_2)}{|X_{Lj}-\mu|^5}.$$

X_{Lj} is the position of any of the L_j, j = 1, 2, 3 on the OX axis.

4. Second order approximation of periodic solution

Let L be any collinear equilibrium points L_j, j = 1, 2, 3. If a new coordinate system is defined with L as origin and L_{x1}, L_{x2}, as axes, parallel to OX and OY as defined by Szebehely [3] respectively, the proper transformation between the two systems is given by the relations

$$X_1 = X_L + x_1, \qquad X_2 = x, \tag{3}$$

Then the system $f_3 - 2nX_4$ and $f_4 + 2nX_3$ are transformed through eqs. (3) in the (x_1, x_2) coordinate system and the equations obtained are expanded by Taylor series upto the second order terms, we get

$$\ddot{x}_1 - 2n\dot{x}_2 = (A_1 + A_1')x_1 + (A_2 + A_2')x_1^2 + (A_3 + A_3')x_2^2,$$

$$\ddot{x}_2 + 2n\dot{x}_1 = (B_1 + B_1')x_2 + (B_2 + B_2')x_1x_2. \tag{4}$$

where

$$A_1 = (1+2R)$$

$$A'_1 = 3(2\sigma_1 - \sigma_2)\left[\frac{1}{2} + \frac{3\mu}{|X_{Lj}+1-\mu|^5}\right] + 3(2\sigma_1' - \sigma_2')\left[\frac{1}{2} + \frac{3(1-\mu)}{|X_{Lj}-\mu|^5}\right],$$

$$A_2 = -3\left[\frac{(1-\mu)(X_{Lj}-\mu)}{|X_{Lj}-\mu|^5} + \frac{\mu(X_{Lj}+1-\mu)}{|X_{Lj}+1-\mu|^5}\right],$$

$$A_2' = \frac{-15\mu(2\sigma_1 - \sigma_2)(X_{Lj}+1-\mu)}{|X_{Lj}-\mu+1|^7} - \frac{15(1-\mu)(2\sigma_1'-\sigma_2')(X_{Lj}-\mu)}{|X_{Lj}-\mu|^7},$$

$$A_3 = -\frac{A_2}{2},$$

$$A_3' = \frac{A_2'}{4} - \frac{15\mu(\sigma_1 - \sigma_2)(X_{Lj}+1-\mu)}{2|X_{Lj}+1-\mu|^7} - \frac{15(1-\mu)(\sigma_1'-\sigma_2')(X_{Lj}-\mu)}{2|X_{Lj}-\mu|^7},$$

$$B_1' = \frac{3}{2}(2\sigma_1 - \sigma_2)\left[1 - \frac{\mu}{|X_{Lj}-\mu+1|^5}\right] - \frac{3\mu(\sigma_1-\sigma_2)}{|X_{Lj}+1-\mu|^5} + \frac{3}{2}(2\sigma_1'-\sigma_2')\left[1 - \frac{(1-\mu)}{|X_{Lj}-\mu|^5}\right] - \frac{3(1-\mu)(\sigma_1'-\sigma_2')}{|X_{Lj}-\mu|^5},$$

$$B_2 = -A_2,$$

$$B_2' = -A_2' + \frac{15\mu(\sigma_1-\sigma_2)(X_{Lj}+1-\mu)}{|X_{Lj}-\mu+1|^7} + \frac{15(1-\mu)(\sigma_1'-\sigma_2')(X_{Lj}-\mu)}{|X_{Lj}-\mu|^7}.$$

We search for periodic solutions in the form of second order expansions in powers of parameter ε

$$x_1(\tau) = x_{11}(\tau)\varepsilon + x_{12}(\tau)\varepsilon^2, \qquad x_2(\tau) = x_{21}(\tau)\varepsilon + x_{22}(\tau)\varepsilon^2. \tag{5}$$

In order to erase any secular term in the future analysis, we substitute the relations (5) into (4). Retaining terms of powers in ε not greater than two and denoting by dot (.) the τ-derivatives and ignoring the terms

$\sigma_i\varepsilon^2, \sigma_i'\varepsilon^2, (i=1, 2)$, we have to solve the system :

$$
\begin{aligned}
&(\ddot{x}_{11}\varepsilon + \ddot{x}_{12}\varepsilon^2) - 2n(\dot{x}_{21}\varepsilon + \dot{x}_{22}\varepsilon^2) \\
&= (A_1 + A_1')(x_{11}\varepsilon + x_{12}\varepsilon^2) \\
&\quad + (A_2 + A_2')(x_{11}\varepsilon + x_{12}\varepsilon^2)^2 \\
&\quad + (A_3 + A_3')(x_{21}\varepsilon + x_{22}\varepsilon^2)^2, \\
&(\ddot{x}_{21}\varepsilon + \ddot{x}_{22}\varepsilon^2) + 2n(\dot{x}_{11}\varepsilon + \dot{x}_{12}\varepsilon^2) \\
&= (B_1 + B_1')(x_{21}\varepsilon + x_{22}\varepsilon^2) \\
&\quad + (B_2 + B_2')(x_{11}\varepsilon + x_{12}\varepsilon^2)(x_{21}\varepsilon + x_{22}\varepsilon^2),
\end{aligned} \tag{6}
$$

where

$$g_1(\tau) = A_2 x_{11}^2 + A_3 x_{21}^2,$$

$$g_2(\tau) = B_2 x_{11} x_{21}.$$

Defining the differential operator

$$F_1(D) = \begin{pmatrix} D^2 - (A_1 + A_1') & -2nD \\ 2nD & D^2 - (B_1 + B_1') \end{pmatrix}$$

we shall solve the eqs. (6) by equating the coefficient of the same powers of ε.

4.1. *The first order system* :

$$F_1(D)\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

The general solution of the eqs. (7) is

$$
\begin{aligned}
x_{11}(\tau) &= \sum_{i=1}^{4} c_i \exp(\lambda_i \tau), \\
x_{21}(\tau) &= \sum^{4} d_i \exp(\lambda_i \tau),
\end{aligned} \tag{8}
$$

where λ_i i = 1, 2, 3, 4. are the characteristic roots of the system (7). By a suitable choice of the coefficients of the exponential terms of eq. (8), we may have a special periodic solution, which contains only the frequency corresponding to a specific imaginary part. We denote this frequency by ω. Eqs. (7) admit the periodic solution

$$x_{11}(\tau) = A\cos(\omega\tau) + B\sin(\omega\tau),$$

$$x_{21}(\tau) = A^*\cos(\omega\tau) + B^*\sin(\omega\tau),$$

where the coefficients A, B, A^*, B^* are connected by the relations

$$A = A_c + A_\sigma,$$

$$A^* = \frac{2n\omega B}{B_1 + B_1' + \omega^2},$$

$$B^* = B_c^* + B_\sigma^*,$$

where :

$$B_c^* = -\frac{2A_c}{B_1 + \omega_c^2}\omega_c,$$

$$B_\sigma^* = -\frac{2A_c}{B_1 + \omega_c^2}\omega_\sigma - \frac{\omega_c(B_1' + 2\omega_c\omega_\sigma)}{B_1 + \omega_c^2}$$

$$+\frac{3}{4}(2\sigma_1 - \sigma_2)\omega_c + \frac{3}{4}(2\sigma_1' - \sigma_2')\omega_c \Big] - \frac{2A_\sigma\omega_c}{B_1 + \omega_c^2},$$

$$A_c = \frac{1}{A_2}\left[-A_1 - \omega_c^2 + \frac{4\omega_c^2}{B_1 + \omega_c^2}\right],$$

$$A_\sigma = \frac{1}{A_2}\Big[-A_1' - 2\omega_c\omega_\sigma + \frac{4}{B_1 + \omega_c^2}$$

$$\Big\{2\omega_c\omega_\sigma + \frac{3}{2}(2\sigma_1 - \sigma_2)\omega_c^2 + \frac{3}{2}(2\sigma_1' - \sigma_2')\omega_c^2$$

$$\frac{\omega_c^2(B_1' + 2\omega_c\omega_\sigma)}{B_1 + \omega_c^2} + \frac{A_2'}{A_2^2}\Big(A_1 + \omega_c^2 - \frac{4\omega_c^2}{B_1 + \omega_c^2}$$

Without any loss of generality, we put $x_{21}(0) = 0$. Then $A^* = 0$ and consequently $B = 0$. This means that $x_{11}'(0) = 0$. Finally, the above solution becomes

$$
\begin{aligned}
x_{11}(\tau) &= A\cos(\omega\tau), \\
x_{21}(\tau) &= B^*\sin(\omega\tau).
\end{aligned} \tag{9}
$$

4.2. *The second order system* :

$$F_2(D)\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} g_1(\tau) \\ g_2(\tau) \end{pmatrix}, \tag{10}$$

where

$$F_2(D) = \begin{pmatrix} D^2 - A_1 & -2nD \\ 2nD & D^2 - B_1 \end{pmatrix}$$

Substituting eqs. (9) into eqs. (10), functions g_i, i = 1, 2 become

$$g_1(\tau) = K_0 + K_1\cos(2\omega\tau),$$

$$g_2(\tau) = \Lambda_1\sin(2\omega\tau),$$

where

$$K_0 = \frac{1}{2}\left[A_2 A_c^2 + A_3 B_c^{*2} \right],$$

$$K_1 = \frac{1}{2}\left[A_2 A_c^2 - A_3 B_c^{*2} \right],$$

$$\Lambda_1 = \frac{1}{2}\left[B_2 A_c B_c^{*} \right].$$

A periodic solution of system (10) is

$$\begin{aligned} x_{12}(\tau) &= M_0 + M_1 \cos(2\omega\tau), \\ x_{22}(\tau) &= N_1 \sin(2\omega\tau), \end{aligned} \tag{11}$$

where

$$M_0 = -\frac{K_0}{A_1},$$

$$M_1 = \frac{1}{\psi}\left[-4K_1\omega_c^2 + 4\Lambda_1\omega_c - B_1 K_1 \right],$$

$$N_1 = \frac{1}{\psi}\left[-4\Lambda_1\omega_c^2 + 4K_1\omega_c - A_1\Lambda_1 \right],$$

$$\psi = 16\omega_c^4 - 4\left[4 - A_1 - B_1\right]\omega_c^2 + A_1 B_1.$$

Finally, a second order approximation of the periodic solution around the collinear equilibrium points, as a function of parameter ε, is obtained from eqs. (9) and (11) as

$$\begin{aligned} x_1(\tau,\varepsilon) &= \left[A\cos(\omega\tau)\right]\varepsilon + \left[M_0 + M_1\cos(2\omega\tau)\right]\varepsilon^2, \\ x_2(\tau,\varepsilon) &= \left[B^{*}\sin(\omega\tau)\right]\varepsilon + \left[N_1\sin(2\omega\tau)\right]\varepsilon^2 \end{aligned} \tag{12}$$

Period of this solution is

$$T = \frac{2\pi}{\omega}.$$

5. Numerical results (μ = 0.01215)

Using the formulae (12), we find a first approximation which are close to L_j, j = 1, 2, 3. Then, by a linear predictor-corrector algorithm based on numerical integration of the equations of motion (1) and first order variations eqs. (2), we computed the periodic orbits by taking different values of semi-axes of the triaxial rigid bodies in the earth-moon system. The Liapunov stability of each periodic solution is also examined. The stability parameters are calculated (approximately) by means of additional numerical integrations of the equations of motion for which the knowledge of the stability parameter 'a' is sufficient. (For stability : $|a|<1$ Markellos [2]).

The results for each family are represented in tabular and graphical form. The results of the families A, B, and C are given in Tables 3, 4 and 5 respectively. The results are graphically shown in Figures 1, 2 and 3.

Table 3. Periodic orbits around L_2 (Family A).

Case	1	2	3	4	5
X_0	−0.8369180067837486	−0.8369163551787104	−0.836914708421407	−0.8369130663936077	−0.8369114290394294
$\dot{Y}_0$	$-4.463895215162087 \times 10^{-9}$	$-1.385902000019717 \times 10^{-9}$	$-2.84721804580216 \times 10^{-9}$	$7.768771279485732 \times 10^{-10}$	$1.281908759658305 \times 10^{-9}$
T	2.6915848174	2.691423256573832	2.691448591261801.	2.69159370531843	2.6916903674
a	$-1.168460868400305 \times 10^{11}$	$-3.762101428707008 \times 10^{11}$	$-1.83153475583645 \times 10^{11}$	$6.715973492310867 \times 10^{11}$	$4.071624558188485 \times 10^{11}$
b	$4.139404274014810 \times 10^{2}$	$4.137892871735222 \times 10^{2}$	$4.138647955887082 \times 10^{2}$	$4.140556182643960 \times 10^{2}$	$4.142475555097498 \times 10^{2}$
c	$3.303237828577061 \times 10^{19}$	$3.425564428543048 \times 10^{20}$	$8.11749115384392 \times 10^{19}$	$2.090878685351613 \times 10^{21}$	$4.00795838050953 \times 10^{20}$
d	$-1.168460868400500 \times 10^{11}$	$-3.762101428711530 \times 10^{11}$	$.1.83153475583977 \times 10^{11}$	$6.715973492312582 \times 10^{11}$	$4.071624558199573 \times 10^{11}$

Table 4. Periodic orbits around L_3 (Family B).

Case	1	2	3	4	5
X_0	1.005062401872425	1.005062143818756	1.00506188655168	1.005061629943756	1.005061374059814
$\dot{Y}_0$	$-1.049539117174321 \times 10^{-10}$	$-1.496995186805439 \times 10^{-10}$	$-8.5870004785839 \times 10^{-11}$	$-5.726803819819105 \times 10^{-11}$	$1.040739600460060 \times 10^{-10}$
T	6.2183934845	6.2183885117	6.2183838369	6.218006790169993	6.2183747659
a	5.520879594627006	$-1.168977046557862 \times 10^{3}$	$-4.0840417362227 \times 10^{3}$	$-9.181256040350725 \times 10^{3}$	$6.738850681005347 \times 10^{3}$
b	$3.043002796331533 \times 10^{-1}$	$3.043034758233631 \times 10^{-1}$	$3.04306936564127 \times 10^{-1}$	$3.039278798290422 \times 10^{-1}$	$3.043140314461915 \times 10^{-1}$
c	$2.656728107860189 \times 10^{2}$	[illegible]779232826919565 $\times 10^{7}$	$2.169001956856698 \times 10^{8}$	$1.096129923334235 \times 10^{9}$	$5.900482044359133 \times 10^{8}$
[illegible]	5.520879594609950	[illegible]8977047822943 × [illegible]	$-4.08404137874119 \times 10^{3}$	$-9.181256044144799 \times 10^{3}$	$6.738850675986001 \times 10^{3}$

Table 5. Periodic orbits around L_1 (Family C).

Case	1	2	3	4	5
X_0	-1.155679910921179	-1.155681389414960	-1.155682859439477	-1.155684327145527	-1.155685792703119
$\dot{Y}_0$	$-1.1789183476925273 \times 10^{-8}$	$1.892201129843619 \times 10^{-9}$	$-4.318869089432017 \times 10^{-9}$	$-3.200299573045917 \times 10^{-9}$	$1.064987974675203 \times 10^{-8}$
l	3.373154424512284	3.3732872766	3.372488330213614	3.373278683943536	3.372316290004429
a	$-3.499083897177702 \times 10^{10}$	$2.181367720850140 \times 10^{11}$	$-9.541390500434169 \times 10^{10}$	$-1.289937237076567 \times 10^{11}$	$3.868533391090335 \times 10^{10}$
b	$2.836347602147766 \times 10^{2}$	$2.837435430600598 \times 10^{2}$	$2.832817258006506 \times 10^{2}$	$2.837928368684447 \times 10^{2}$	$2.832307917860620 \times 10^{2}$
c	$4.328576628832052 \times 10^{18}$	$1.681614362731562 \times 10^{20}$	$3.222590849944498 \times 10^{19}$	$5.879358901495340 \times 10^{19}$	$5.298506336431023 \times 10^{18}$
d	$-3.499083897159017 \times 10^{10}$	$2.181367720840606 \times 10^{11}$	$-9.541390500277963 \times 10^{10}$	$-1.289937237078810 \times 10^{11}$	$3.868533391274262 \times 10^{10}$

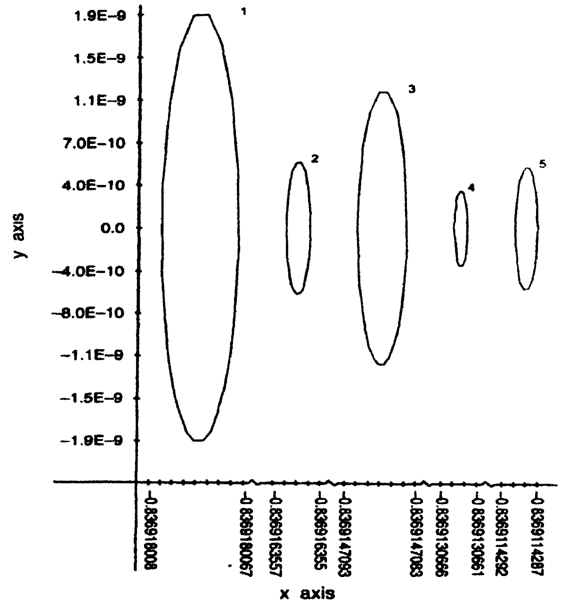

Figure 1. Periodic orbits around L_2 (Family A).

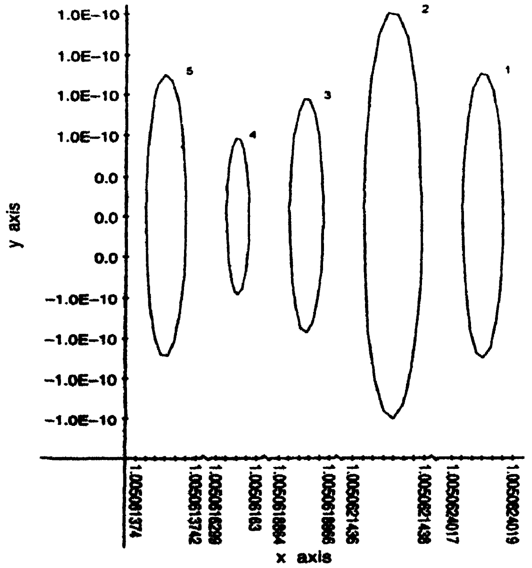

Figure 2. Periodic orbits around L_3 (Family B).

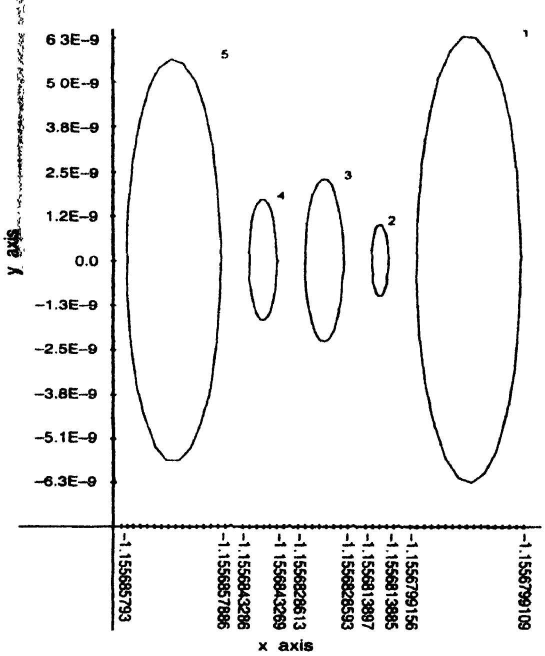

Figure 3. Periodic orbits around L_1 (Family C).

6. Conclusion

With the help of predictor-corrector method, we have computed the initial conditions by taking different values of the semi-axes of the triaxial rigid bodies. With these initial conditions, we have drawn actual periodic orbits in different cases. In the families A and B, we found that as we increase the perturbations in the semi-axes, the periodic orbits move towards the origin. However in family C, we found that as we increase the perturbations in the semi-axes, the periodic orbits move away from the origin. Finally, we have proved that these periodic orbits are unstable because in all cases $|a|>1$.

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